Nonlocal Reaction–Diffusion Models of Wealth Distribution

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Mathematical Ecology   Nonlinear Dynamics   Pattern Formation

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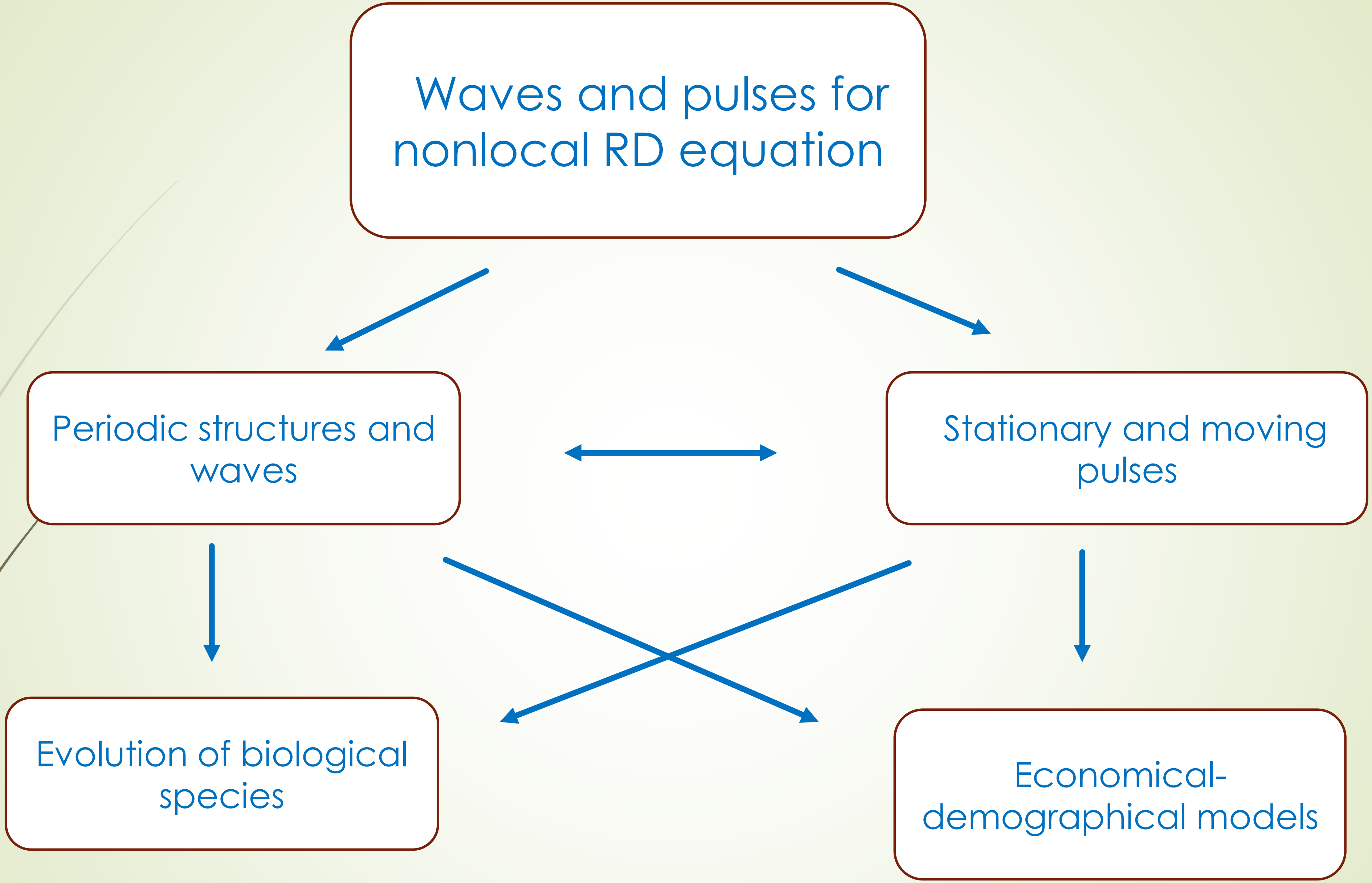
Mathematical Ecology   Biological Invasions   Spatial Ecology


Waves and pulses for nonlocal RD equation
Periodic structures and waves
Stationary and moving pulses
Evolution of biological species
Economical-demographical models

Nonlocal reaction-diffusion equation

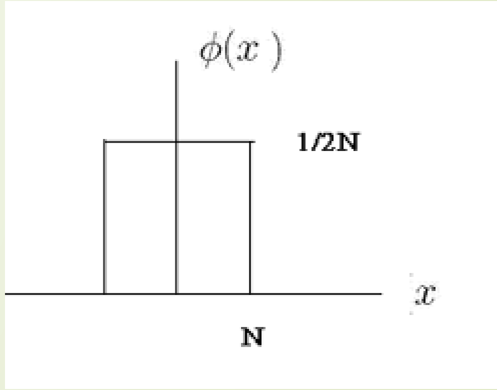
$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^k(1 - H(u)) - \sigma u$$

$$H(u) = u$$

local

$$H(u) = \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy$$

nonlocal



$$H(u) = s\int_{-\infty}^{\infty} u(y,t)dy$$

global

Periodic waves and patterns in nonlocal RD equation

Stability analysis – Pattern formation

Britton, Gourley, …

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + ku\left(1 - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy\right)$$

Homogeneous in space stationary solutions: $u = 0$, $u = 1$

Spectrum of the linearized operator

$$du'' - \sigma \int_{-\infty}^{\infty} \phi(x-y)u(y)dy = \lambda u$$

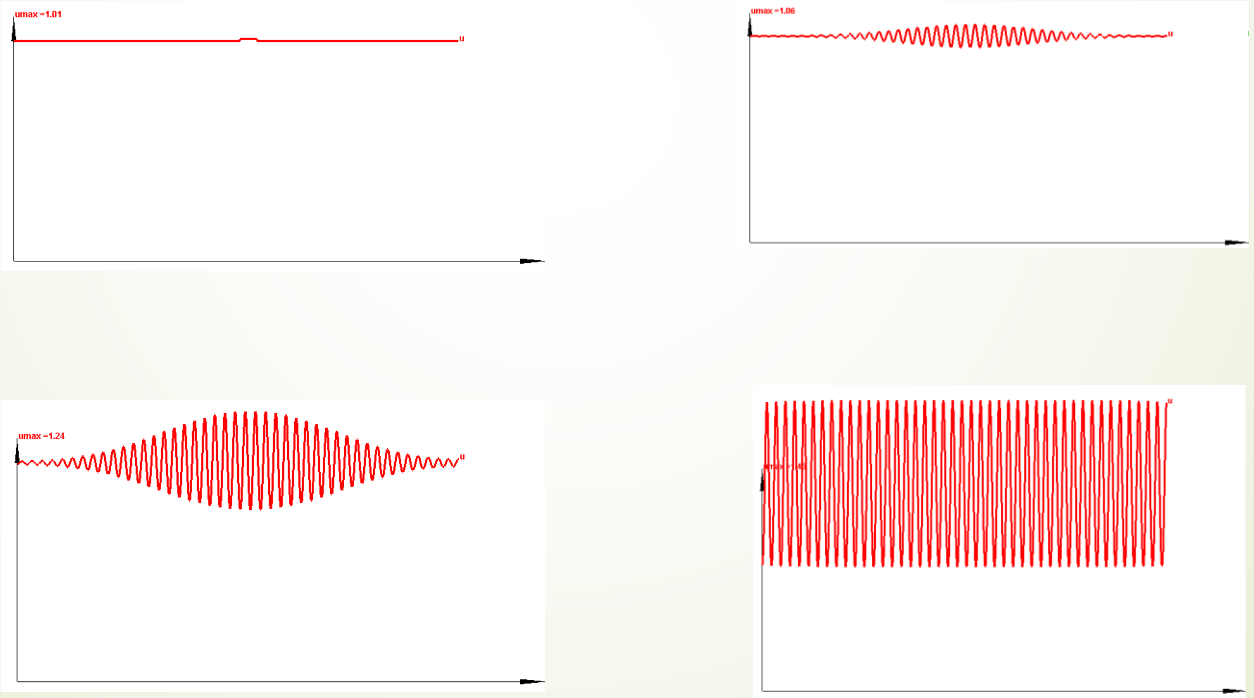
Fourier transform

$$-\lambda = d\xi^2 + \frac{\sigma}{\xi N}\,\sin(\xi N)$$

Instability condition: d/(σ N^2) < const

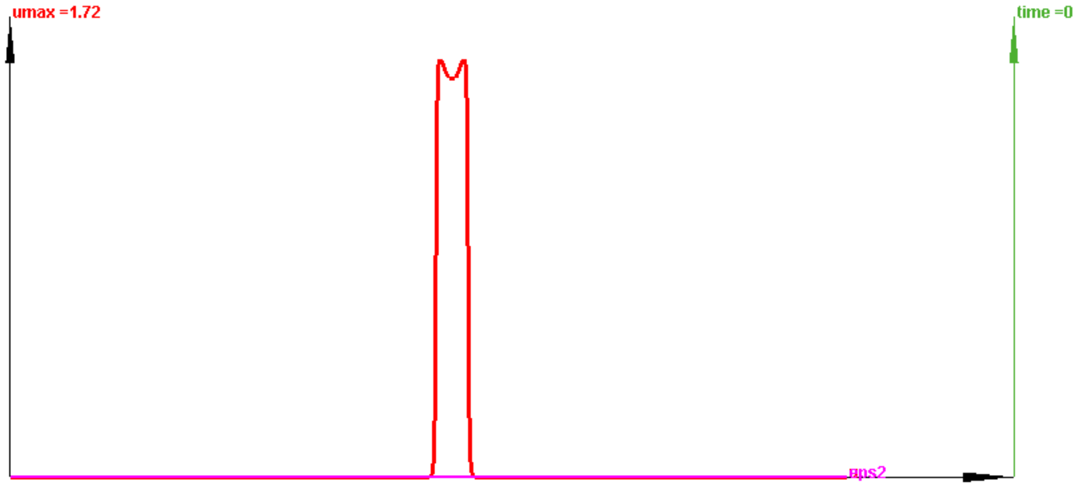
Bifurcation of periodic structures

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + ku\left(1 - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy\right)$$

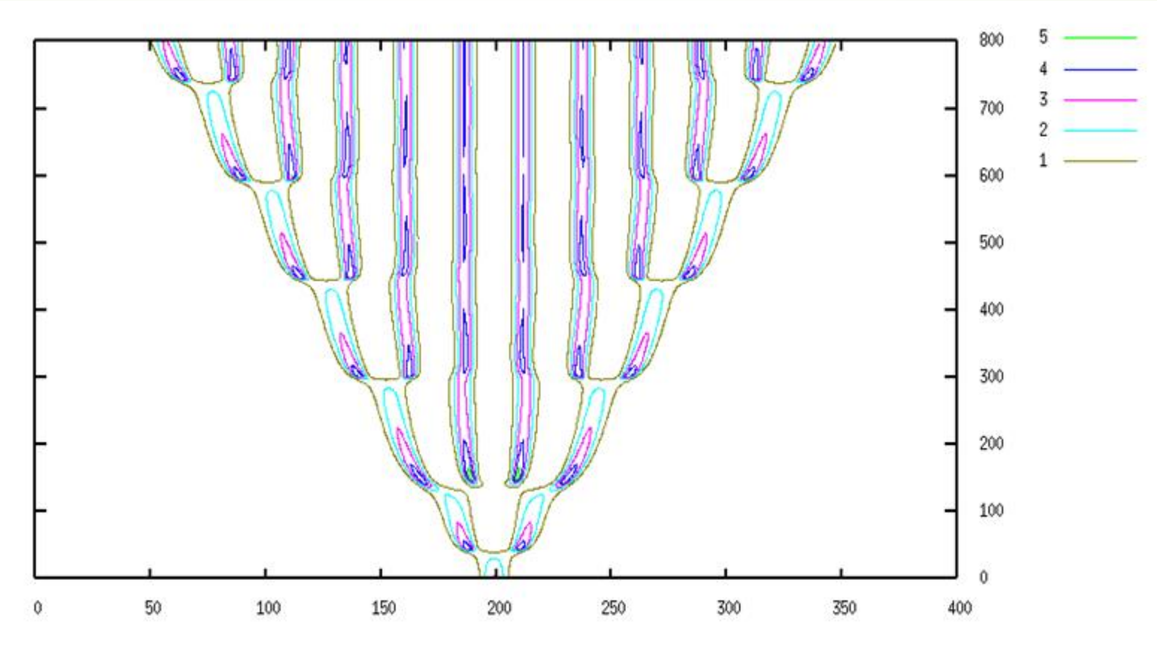


Nonlocal consumption: periodic wave

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + ku\left(1 - \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy\right)$$



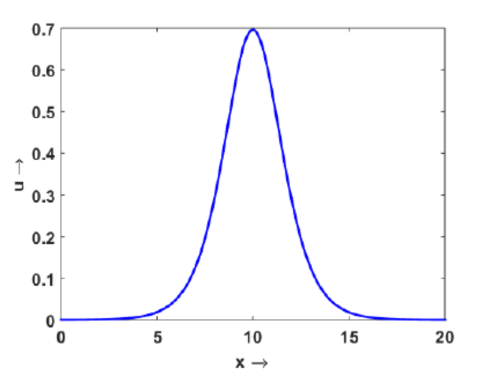
Level lines of the solution



Stationary and moving pulses

Global consumption: pulses

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^k\left(1 - s\int_{-\infty}^{\infty} u(y,t)dy\right) - \sigma u$$



Local consumption (bistable nonlinearity)

- pulse exists, unstable

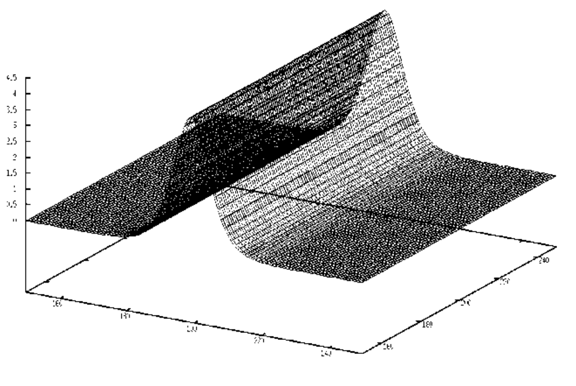
Global consumption

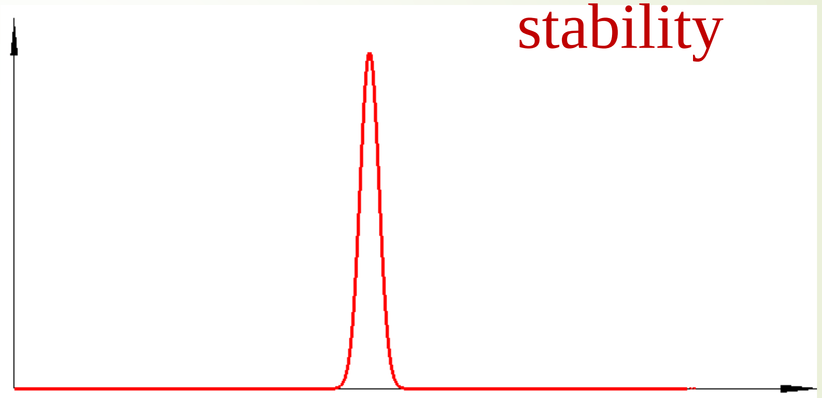
- two pulses exist, one of them is stable

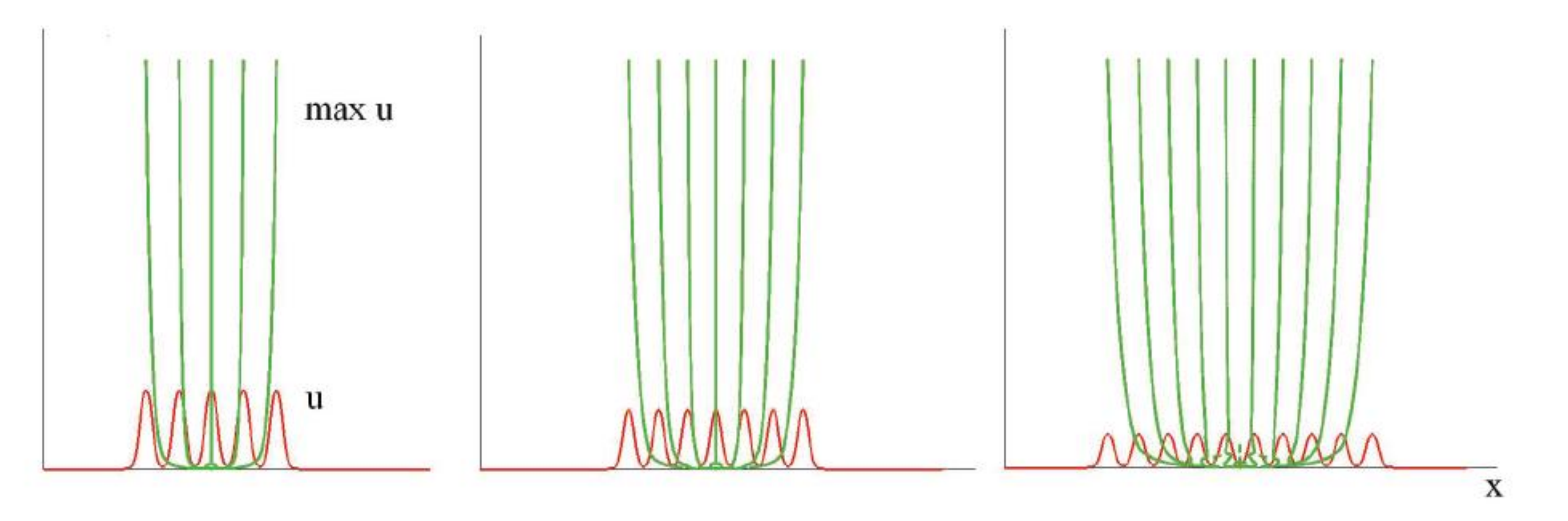
Single and multiple pulses

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + a u^k \left(1 - s\int_{-\infty}^{\infty} u(y,t)dy\right) - \sigma u$$

Existence and stability







Bifurcation of pulses

Nonlocal problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u^2(1 - I(u)) - \sigma u$$

$$I(u) = \int_{-L}^{L} u(x,t)dx$$

Local problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u^2(1 - Lu) - \sigma u,$$

$$0 < x < L \qquad x = 0, L : \frac{\partial u}{\partial x} = 0$$

Stationary solutions

$$u(1 - Lu) = \sigma$$

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Doubly nonlocal reaction–diffusion equations and the emergence of species

M. Banerjee[a,*], V. Vougalter[b], V. Volpert[c]

Linearization

Nonlocal problem

$$Dv'' + \sigma v - u_2^2 I(v) = \lambda v$$

Local problem

$$Dv'' + \sigma v - u_2^2 L v = \lambda v$$

Eigenvalues

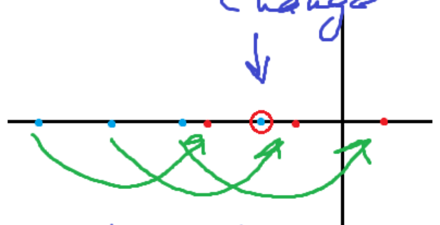
$$\lambda_0 = \sigma - u_2^2 L \ , \quad \lambda_k = \sigma - D(k\pi/L)^2 \ , \quad k = 1, 2, \ldots$$

$$\lambda_k = \sigma - u_2^2 L - D(k\pi/L)^2 \ , \quad k = 1, 2, \ldots$$

Second eigenvalue is positive if

$$\sigma > D\pi^2/L^2$$

λ_0 does not change



λ_k increase

$k = 1, 2, \ldots$

All eigenvalues are negative

Bifurcation of pulses from the second ev !

Global bifurcations

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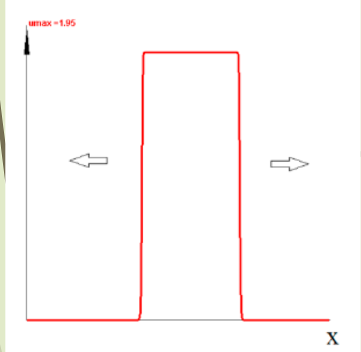
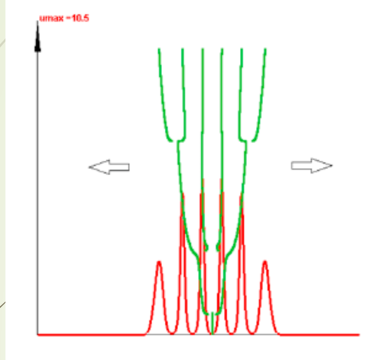
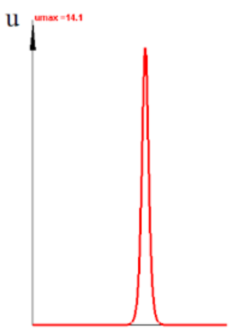
Pulses and waves for a bistable nonlocal reaction–diffusion equation

V. Volpert



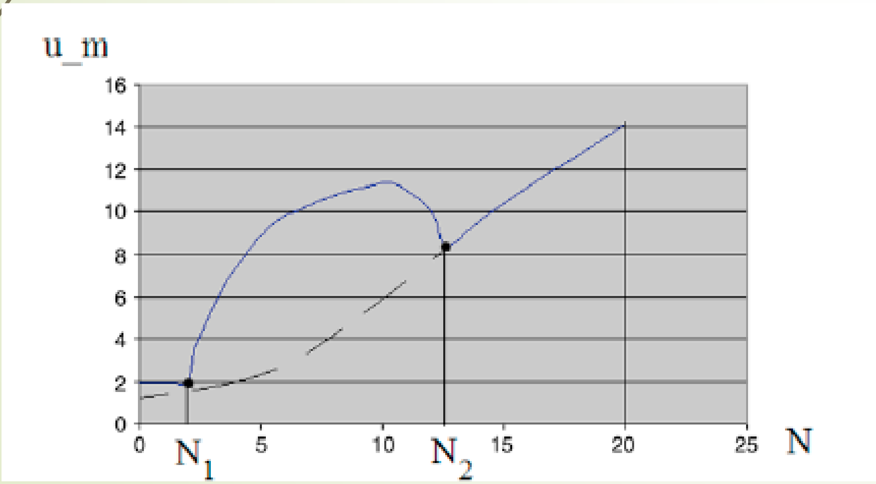


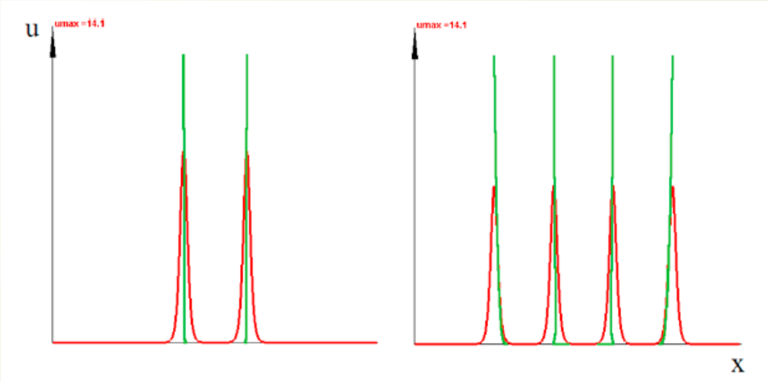


$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + au^2(1 - J(u)) - \sigma u,$$

$$J(u) = \int_{-\infty}^{\infty} \Phi(x-y)u(y,t)dy,$$

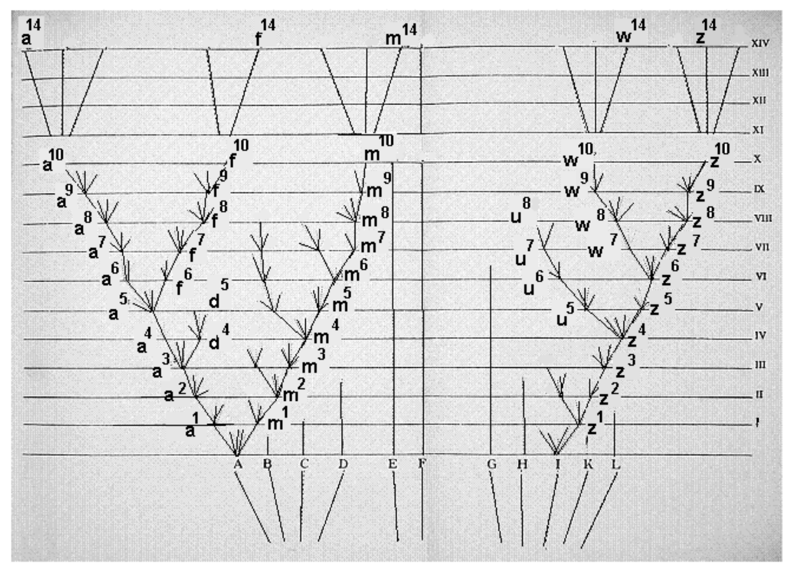
$$\phi(x) = \begin{cases} 1/(2N), & |x| < N \\ 0, & |x| \ge N \end{cases}$$

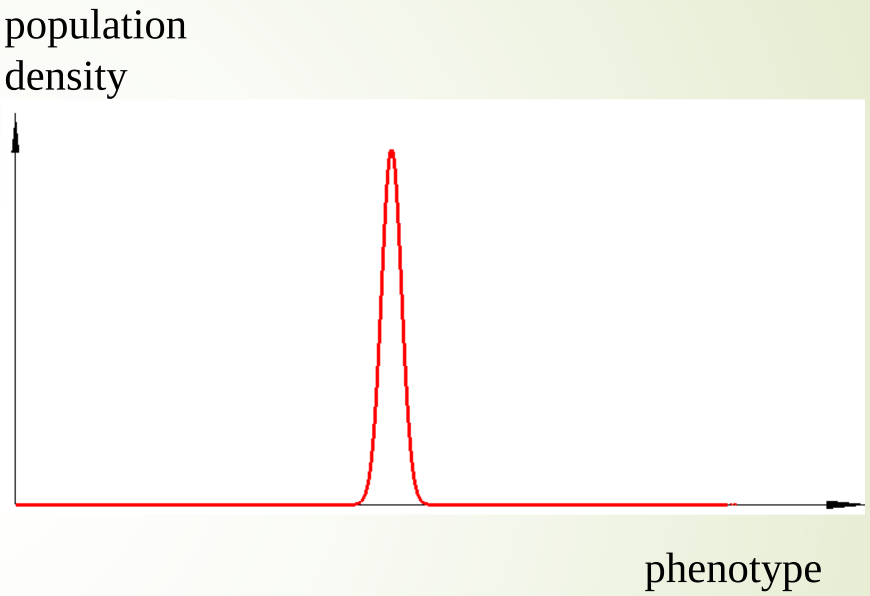




Biological applications

Evolution: Darwin’s diagram and its mathematical interpretation

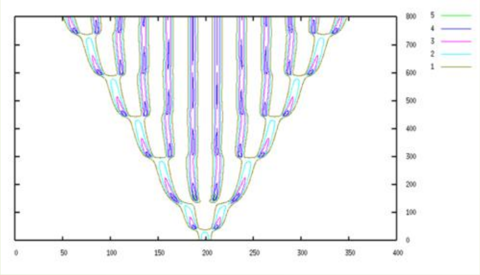


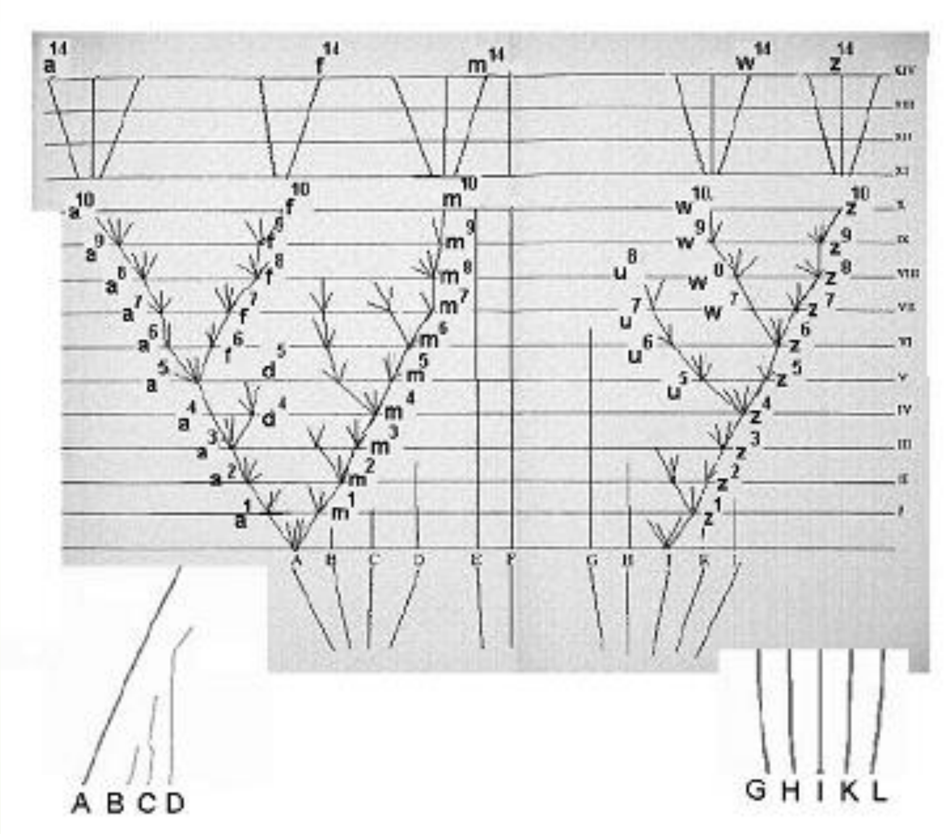


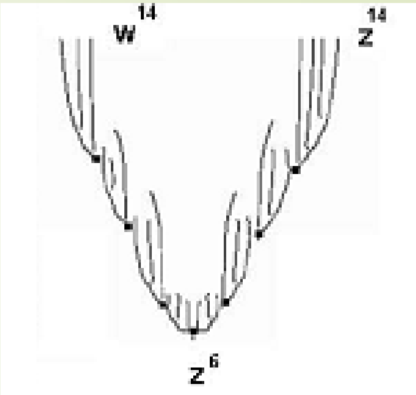
Let A to L represent the species of a genus large in its own country; these species are supposed to resemble each other in unequal degrees, as is so generally the case in nature, and is represented in the diagram by the letters standing at unequal distance ... The little fan of diverging dotted lines of unequal length proceeding from (A), may represent its varying offspring.

Patterns in the diagram:

periodic waves, single and multiple pulses, competition of species, survival of better adapted







Definition of species: Darwin vs Mayr

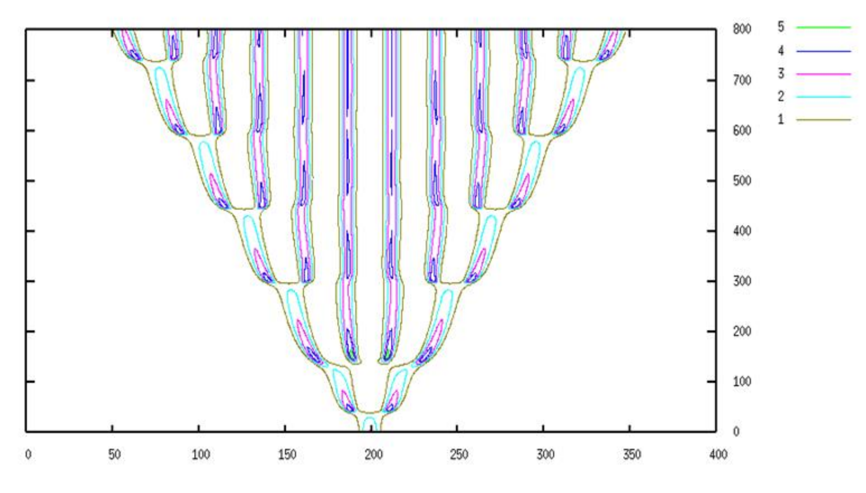
Darwin: morphologically similar individuals
Mayr: reproduction

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + a(S(u))^2(1 - J(u)) - bu,$$

$$S(u) = \frac{1}{2h_1}\int_{-\infty}^{\infty} \psi(x-y)u(y,t)dy, \quad \psi(z) = \begin{cases} 1, & |z| \le h_1 \\ 0, & |z| > h_1 \end{cases},$$

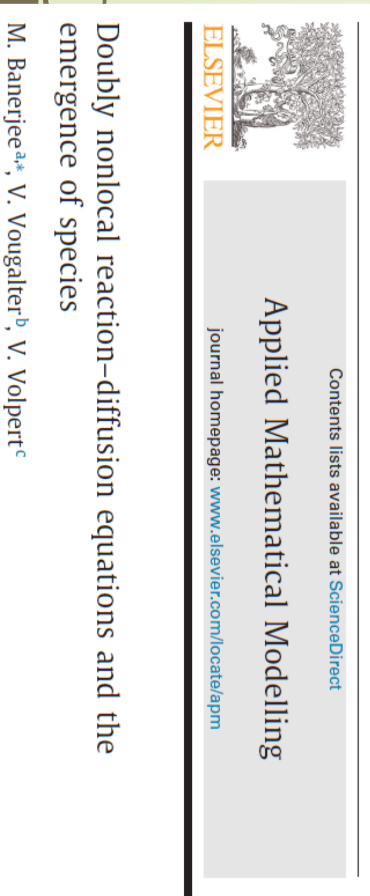
$$J(u) = r(h_2)\int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy, \quad \phi(z) = \begin{cases} 1, & |z| \le h_2 \\ 0, & |z| > h_2 \end{cases}.$$

Phenotypes of parents can be different



Species do not emerge if reproduction of different is allowed (with those who consume the same resources)

Mayr's condition is necessary for Darwin's speciation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Doubly nonlocal reaction–diffusion equations and the emergence of species

M. Banerjee[a,*], V. Vougalter[b], V. Volpert[c]

Demographic-economical models

Wealth-population model

Wealth
population

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + F(u,p),$$

$$\frac{\partial p}{\partial t} = D_p \frac{\partial^2 p}{\partial x^2} + G(u,p),$$

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Interaction of human migration and wealth distribution

V. Volpert[a,b,*], S. Petrovskii[c], A. Zincenko[c]

$$F(u,p) = W(u,p) - S(u,p),$$

$$S(u,p) = au + (r + su)p.$$

$$W = bH^{\nu}Q^{\beta}M^{\gamma},$$

$$W(u,p) = h(u)g(p)M$$

$$h(u) = \frac{a_1 u}{u + c_1}, \quad g(p) = \frac{p}{p + c_2},$$

$$M_1(u) = (\mu + \theta u)(1 - ku).$$

$$M_1(u) = (\mu + \theta u)(1 + \kappa u - ku)$$

$$M_2(u, J(u)) = (\mu + \theta u)(1 - kJ(u)),$$

$$\int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy,$$

$$M_3(u, I(u)) = (\mu + \theta u)(1 - kI(u)),$$

$$\int_{-\infty}^{\infty} u(y,t)dy.$$

Wealth-population model

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wealth production

$$F(u,p) = W(u,p) - S(u,p),$$

wealth consumption

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Wealth-population model

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labor

capital

resources

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production and consumption of resourses

Wealth-population model

cross diffusion

wealth

population

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + F(u,p),$$

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$$\int_{-\infty}^{\infty} u(y,t)dy.$$

production and consumption of resources

Nonlocal consumption of resources

mathematics

Article

Nonlocal Reaction–Diffusion Models of Heterogeneous Wealth Distribution

Malay Banerjee[1], Sergei V. Petrovskii[2] and Vitaly Volpert[3,4,5,*]

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u)p\left(1 - k_2 J(u)\right) - k_3 u,$$

$$\frac{\partial p}{\partial t} = D_p \frac{\partial^2 p}{\partial x^2} + \alpha p\left(\psi(u) - p\right) - \sigma_0 p$$

$$f(u) = \frac{u(b+u)}{1+k_1 u}, \quad \psi(u) = \frac{a_2 u}{u^2 + c_2^2} + \frac{\sigma_1 u}{\alpha(1+u)} + \psi_0 .$$

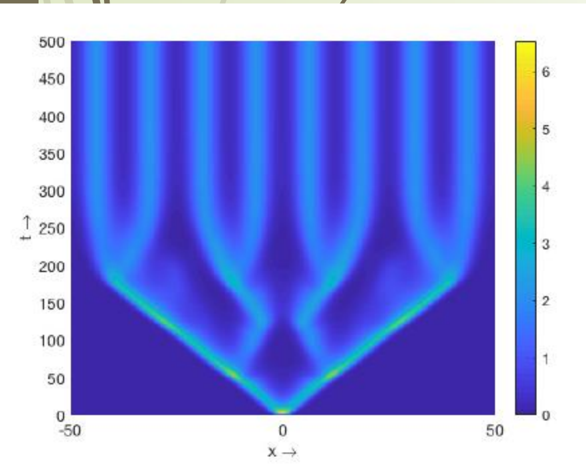
$$J(u) = \int_{-\infty}^{\infty} \varphi(x-y)\, u(y,t)\, dy$$

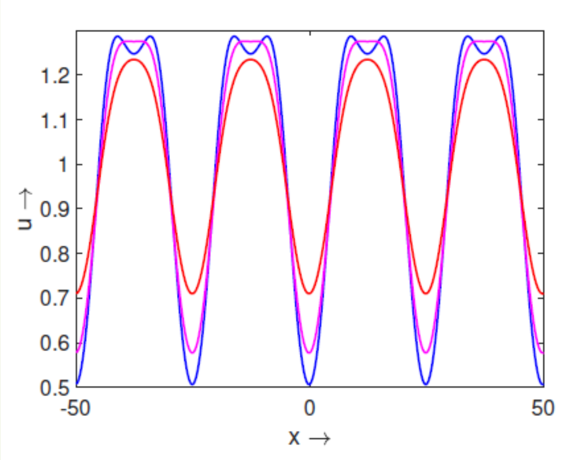
Constant human resources (single equation)

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + F(u,p),$$

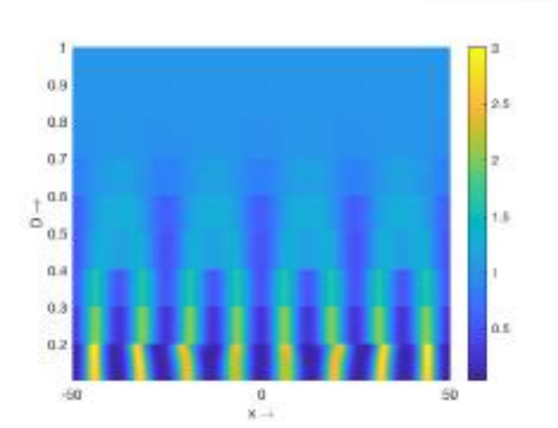
$$F(u,p) = W(u,p) - S(u,p),$$

$$W(u,p) = \frac{a_1 u p}{(u+c_1)(p+c_2)} M, \quad S(u,p) = au + (s+ru)p,$$

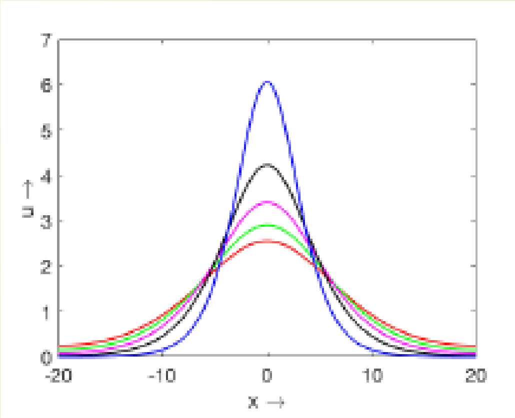




Periodic waves and structures



Bifurcation diagram

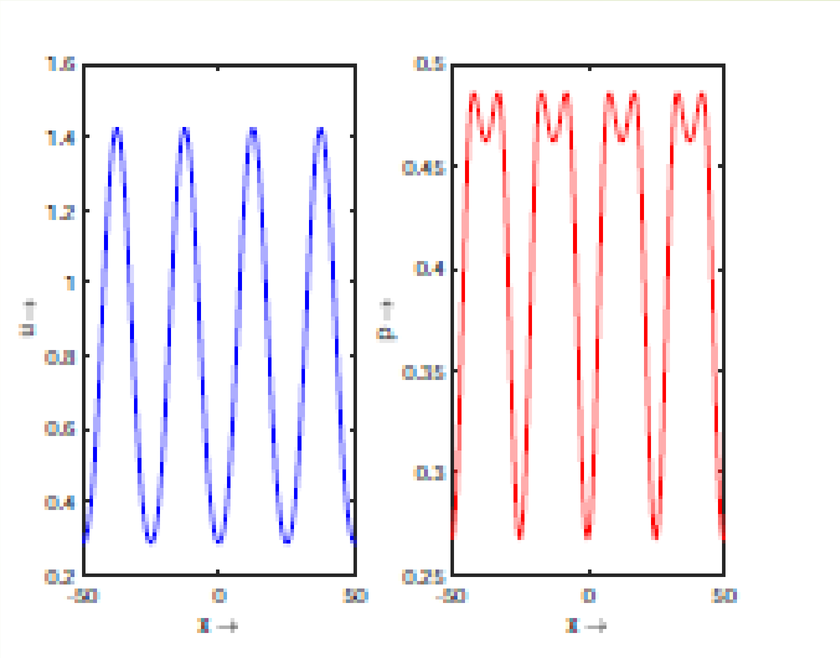


Pulses

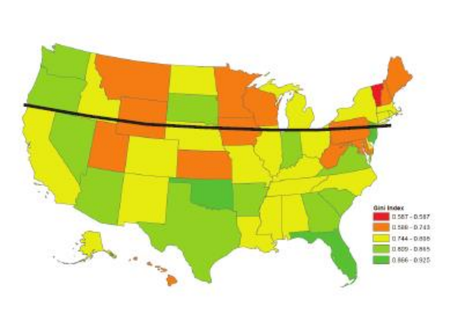
Nonlocal economy (consumption of resources)

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u)p\left(1 - k_2 J(u)\right) - k_3 u,$$

$$\frac{\partial p}{\partial t} = D_p \frac{\partial^2 p}{\partial x^2} + \alpha p\left(\psi(u) - p\right) - \sigma_0 p$$



$$f(u) = \frac{u(b+u)}{1+k_1 u}, \quad \psi(u) = \frac{a_2 u}{u^2 + c_2^2} + \frac{\sigma_1 u}{\alpha(1+u)} + \psi_0 .$$



Nonuniform distribution emerges if D_u is small enough

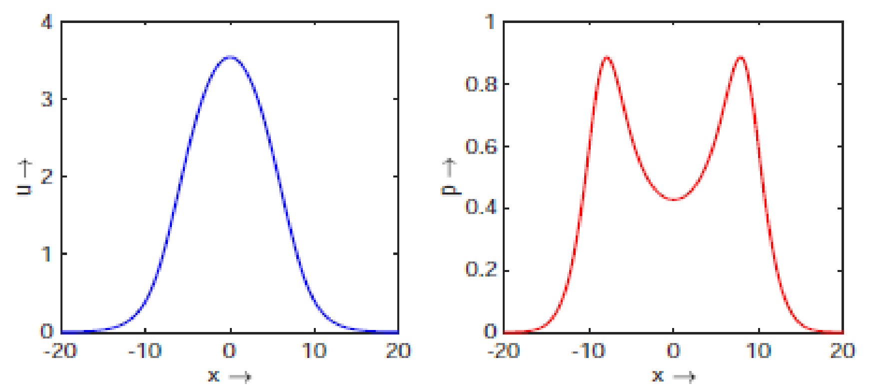
Global economy (consumption of resources)

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u)p\left(1 - k_2 I(u)\right) - k_3 u,$$

$$\frac{\partial p}{\partial t} = D_p \frac{\partial^2 p}{\partial x^2} + \alpha p\left(\psi(u) - p\right) - \sigma_0 p,$$

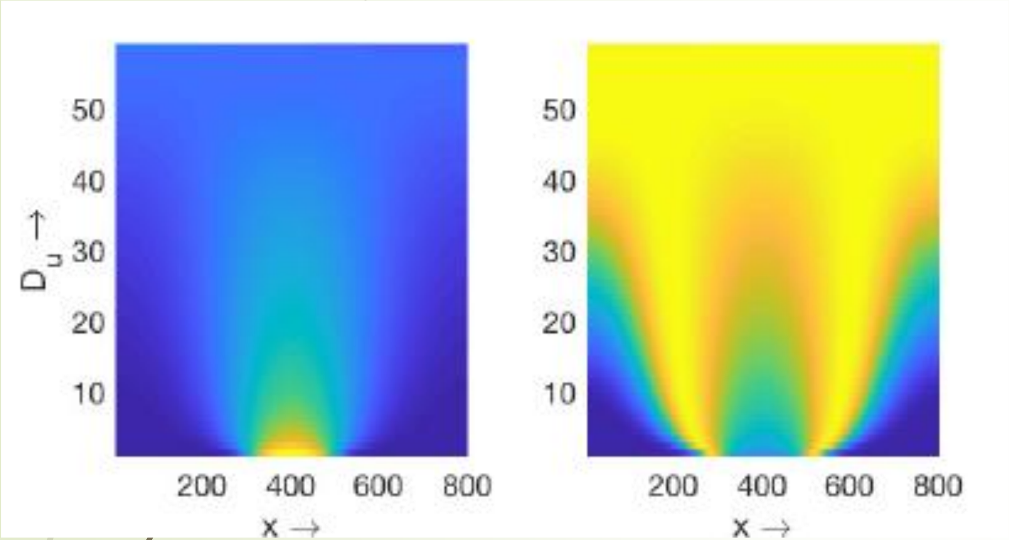
$$f(u) = \frac{u(b+u)}{1+k_1 u}, \quad \psi(u) = \frac{a_2 u}{u^2 + c_2^2} + \frac{\sigma_1 u}{\alpha(1+u)} .$$

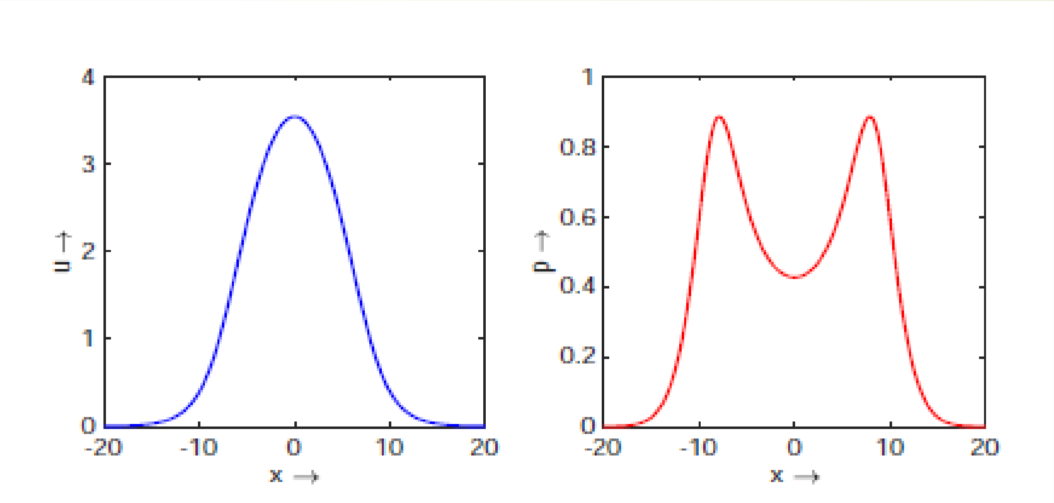
$$I(u) = \int_{-\infty}^{\infty} u(y,t)\, dy$$



Bimodal population distribution

Global economy (consumption of resources)





Nonuniform distribution appears for small Du (redistributions of wealth)

Total wealth and population

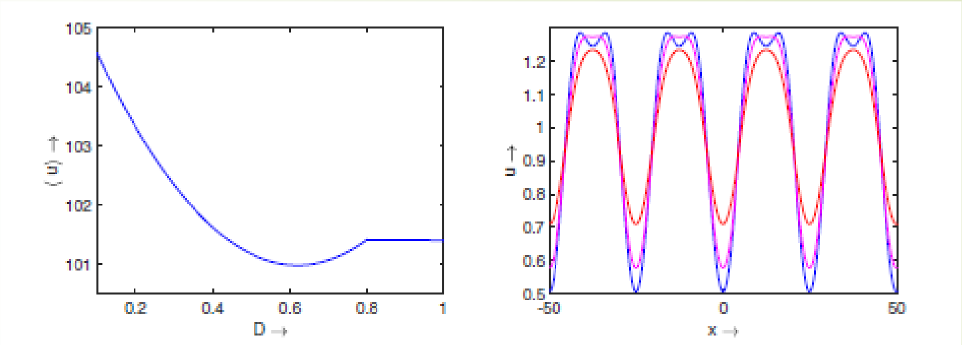
Decrease Du (redistribution of wealth:

- Total wealth increases
- Total population decreases

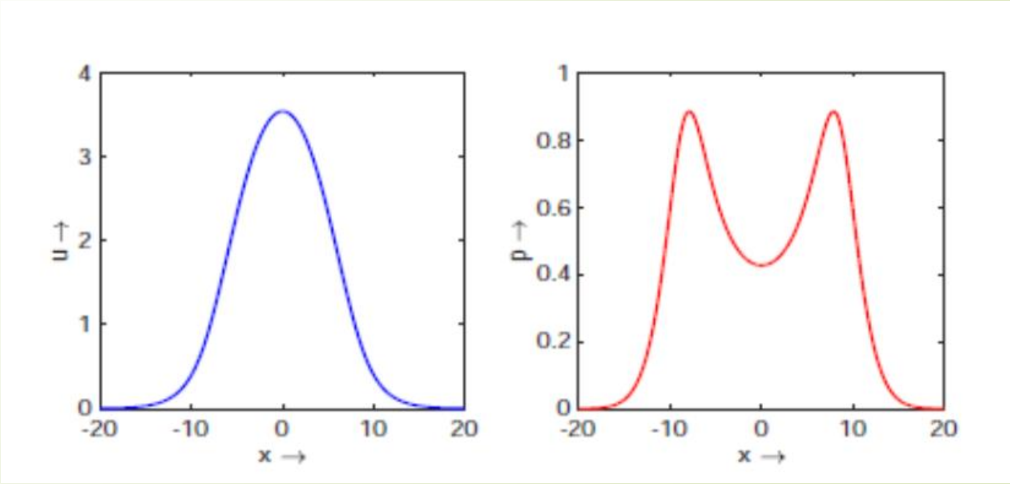
Decrease Dp (migration of population)

- Total wealth increases
- Total population increases

Total wealth



Wealth and population distributions



Conclusions

- Nonuniform wealth distribution emerges if wealth redistribution is small enough
- Periodic structures in the case of nonlocal economy
- One single center of wealth accumulation in the case of global economy

  (transient period, protective measures)
- Nonuniform wealth distribution increases total wealth

  (comparison of different economies)
- Total population can increase or decreases depending on migration