Stability analysis of a prey-predator population model with nonlinear harvesting rate

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Model formulation

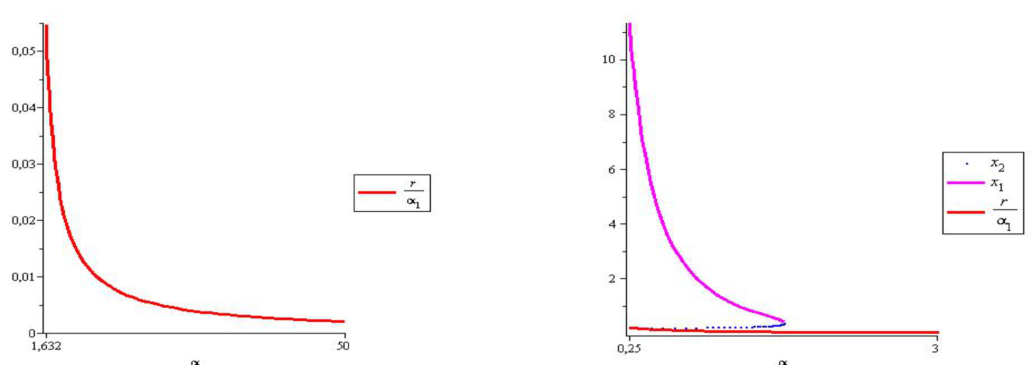
consider the following differential system

$$\begin{cases} \dot{x} &= xr\left(1-\dfrac{x}{k}\right)-axy-\alpha H_1(x) \\ \dot{y} &= caxy-dy-\beta H_2(y) \end{cases} \tag{1}$$

x represents the size of prey population, y the size of predator population. $a, r, k, c, d > 0$; a presents the rate of predation; c denotes the factor of the efficiency of predation which divides a maximum per capita birth rate of the predators into a maximum per capita consumption rate; d is the death rate of the predator; r is the maximum specific growth rate of the prey; k denotes the environmental carrying capacity with which the prey grows logistically in the absence of the predation; α and β are positive parameters related to the harvesting effort.
we suppose that harvesting functions of the prey and the predator in system (1) are respectively, $H_1(x) = x^2$ and $H_2(y) = y^3$.

Bifurcation analysis

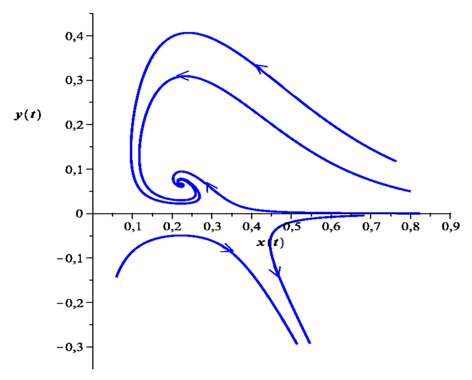
Choosing $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$, we get $d_1 \geq 0$ for $\alpha \geq 0.25$. (x_1, y_1) and (x_2, y_2) extinct after the backward bifurcation value $\alpha = 1.6318$.



The bifurcation diagram for (1) with x_1, x_2, and $\frac{r}{\alpha_1}$ versus α, after the backward bifurcation (the right figure), and the forward bifurcation (the left figure). The two continued lines present the curves of the prey with coexistence equilibrium (x_1, y_1) and $(\frac{r}{\alpha_1}, 0)$, and the dotted line indicates the curve of the prey with coexistence equilibrium (x_2, y_2).

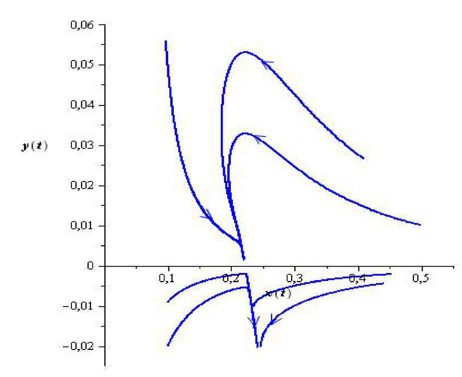
Numerical simulation

Example 1. Setting $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$, $\alpha = 0.1$, we get $d_1 = -0.1 < 0$. The equilibrium (x_2, y_2) exists and is positive and globally asymptotically stable and $(\frac{r}{\alpha_1}, 0)$ is an unstable equilibrium.



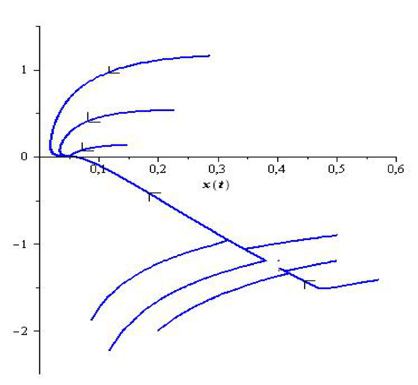
The phase portrait of system (1) when (x_2, y_2) is globally asymptotically stable and $(\dfrac{r}{\alpha_1}, 0)$ is unstable.

Example 2. Setting $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$, $\alpha = 0.25$, we get $d_1 = 0$. The equilibrium (x_2, y_2) coincides with the equilibrium $(\frac{r}{\alpha_1}, 0)$ giving a saddle-node point and a transcritical bifurcation occurs; in this case there is no other positive equilibria.



The phase portrait of system (1) when (x_2, y_2) meets $(\dfrac{r}{\alpha_1}, 0)$ at a saddle-node equilibrium.

Example 3. Taking $d = 0.2$, $r = 0.1$, $k = 0.5$, $c = 1.8$, $a = 0.5$, $\beta = 0.1$ and $\alpha = 1.6318$ we get $d_1 = 0.150868 > 0$. The equilibrium (x_2, y_2) meets the equilibrium (x_1, y_1) giving an unstable one. A saddle-node bifurcation occurs. There is no positive equilibria other than $(\dfrac{r}{\alpha_1}, 0)$ which is stable.



The phase portrait of system (1) when the unstable equilibrium (x_2, y_2) coincides with the stable equilibrium (x_1, y_1).

Preliminary results

We show that solutions of system (1) starting from positive initial conditions are positive and forward bounded. Let

$$x_0 > \max\left(\frac{r}{\alpha_1}, \frac{d}{ca}\right)$$

Then,

$$D = \left\{(x,y) \in \mathbb{R}\times\mathbb{R};\ 0 < x < x_0, 0 < y < \sqrt{\frac{1}{\beta}(cax_0 - d)}\right\}$$

is a positive invariant set for system (1).

Theorem 1

We denote $\alpha_1 = \alpha + \frac{r}{k}$, $d_1 = d - \frac{car}{\alpha_1}$ and $\Delta = \frac{c^2a^4}{\alpha_1^2} - 4\beta d_1$.

Theorem 0.1 *System (1) has the following coexistence equilibria:*
In the case $\beta < \frac{c^2a^4}{4d_1\alpha_1^2}$ and $d_1 > 0$, or in the case $d_1 < 0$, system (1) admits four equilibria: $(0,0)$, $\left(\frac{r}{\alpha_1}, 0\right)$, (x_1, y_1) and (x_2, y_2) where

$$x_1 = \frac{r}{\alpha_1} - \frac{a}{\alpha_1}y_1,\ y_1 = \frac{\frac{-ca^2}{\alpha_1} - \sqrt{\Delta}}{2\beta},$$

$$x_2 = \frac{r}{\alpha_1} - \frac{a}{\alpha_1}y_2 \text{ and } y_2 = \frac{\frac{-ca^2}{\alpha_1} + \sqrt{\Delta}}{2\beta}$$

*If $\beta = \frac{c^2a^4}{4d_1\alpha_1^2}$, system (1) has three equilibria: $(0,0)$, $\left(\frac{r}{\alpha_1}, 0\right)$ and $\left(x_1, \frac{-ca^2}{2\alpha_1\beta}\right)$. When $\beta > \frac{c^2a^4}{4d_1\alpha_1^2}$ and $d_1 > 0$ there exist two equilibria: $(0,0)$ and $\left(\frac{r}{\alpha_1}, 0\right)$. If $d_1 = 0$, system (**??**) admits three equilibria: $(0,0)$, $\left(\frac{r}{\alpha_1}, 0\right)$ and (x_1, y_1).*

Theorem 2

we show that the equilibrium $\left(\frac{r}{\alpha_1}, 0\right)$ is locally unstable, if $d_1 < 0$, and a locally stable when $d_1 > 0$. However, if $d_1 = 0$, then $\left(\frac{r}{\alpha_1}, 0\right)$ is a degenerate equilibrium.

Theorem 0.2 *Equilibrium $(0,0)$ is unstable for system (1). if $d_1 < 0$, equilibrium point $\left(\frac{r}{\alpha_1}, 0\right)$ is a saddle-node, with two hyperbolic sectors and a parabolic one.*
There exists an invariant strong unstable manifold W_u tangent at point $\left(\frac{r}{\alpha_1}, 0\right)$ to the x-axis, on which the behavior of system (1) is repulsive.

Theorem 3

Theorem 0.3 *The equilibrium point (x_1, y_1) is locally stable if $\beta \neq \frac{c^2a^4}{4d_1\alpha_1^2}$ and locally unstable if $\beta = \frac{c^2a^4}{4d_1\alpha_1^2}$. The equilibrium point (x_2, y_2) is locally unstable if $d_1 \geq 0$, and it is either a linear center, a focus or a node if $d_1 < 0$; it is locally stable if $d_1 < 0$ and $\left(1 - \frac{ca}{\alpha_1}\right) \geq 0$.*

Theorem 4

Theorem 0.4 *System (1) does not have a periodic solution and then it does not have a limit cycle in*

$$D^+ = \{(x,y) \in \mathbb{R}\times\mathbb{R};\ x > 0, y > 0\}.$$

Theorem 5

Theorem 0.5 *1) If $d_1 < 0$, then the positive equilibrium point (x_2, y_2) is asymptotically stable; it is globally asymptotically stable in $D \cap D^+$.*
2) If $d_1 > 0$, then the equilibrium point $\left(\frac{r}{\alpha_1}, 0\right)$ is globally asymptotically stable in $D \cap (D^+ \cup (\mathbb{R}_+ \times \{0\}))$; if moreover $\beta > \frac{c^2a^4}{4d_1\alpha_1^2}$, it is globally asymptotically stable in $D \cap (D^+ \cup D^- \cup (\mathbb{R}_+ \times \{0\}))$, where $D^- := \{(x,y) \in \mathbb{R}\times\mathbb{R};\ x > 0, y < 0\}$.
3) If $d_1 < 0$, then the equilibrium point (x_1, y_1) is locally asymptotically stable; it is globally asymptotically stable in D^-.